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SELF-SIMILAR FLOWS BEHIND THE SHOCK WAVES IN A GRAVITATIONAL FIELD

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We consider a self-similar problem of propagation of shock waves in a gravitational field, taking into account the variation of energy with time. The flow is caused by expansion of a spherical piston and the shock waves propagate with constant velocity. For the motions of such a type the velocity, pressure and density can all be expressed as functions of a single, dimensionless parameter. It is established that the energy varies with the Mach number.

Propagation of shock waves during an explosion in a gaseous medium in the presence of gravitational and nongravitational fields was studied by a number of workers. Sedov in [1] studied in detail a self-similar motion of the first kind (according to the classification given in [2]). The similarity index was determined either from the dimensionality concepts, or from the conservation laws. A self-similar problem of motion in a homogeneous medium in a nongravitational field generated by expansion of a plane, spherical or cylindrical surface, was studied by Rogers [3] for the case when the perturbed region was bounded by a strong shock wave. In the perturbed region the total energy increases with time according to the power law.

Below we investigate a self-similar motion in which the total energy grows proportionally to time, in an isothermal gaseous medium, in a gravitational field behind a spherical shock wave moving at a constant speed c_0. The position of the inner flow boundary represented by an expanding surface, is determined by numerical integration of a system of equations for the Mach numbers equal to 1.6 and $\sqrt{10}$.

1. Equations of motion and their solution. Using the spherical polar coordinates r, θ, φ, and placing the origin at the center of an expanding spherical piston, we can write the equations of motion and continuity in the gravitational field, in the form (utilizing the property of spherical symmetry)

$$\frac{dv}{dt}+\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{G\mu}{r^2}=0, \qquad \frac{d\rho}{dt}+\rho\frac{\partial v}{\partial r}+\frac{2\rho v}{r}=0 \tag{1.1}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right)=0, \qquad \frac{\partial\mu}{\partial r}=4\pi\rho r^2$$

$$\left(\frac{d}{dt}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}\right)$$

Here v, p and ρ are the velocity, pressure and density of the medium at the distance r from the coordinate origin, G is the gravitational constant and μ is the mass of the medium contained in the sphere of radius r. In the unperturbed medium the corresponding values of velocity, density, pressure and mass are given by

$$v_1 = 0, \quad \rho_1 = \frac{A}{r^{\omega}}, \quad p_1 = \frac{2\pi A^2 G}{(\omega-1)(3-\omega)}\frac{1}{r^{2\omega-2}} \tag{1.2}$$
$$\mu_1 = \frac{4\pi A}{3-\omega} r^{3-\omega} \qquad (1<\omega<3, \ A = \mathrm{const})$$

At the surface of the shock wave propagating through gas at rest the following conditions must hold:

$$V_2 = \frac{2}{\gamma+1} c_0 (1-q), \qquad \rho_2 = \frac{\gamma+1}{\gamma-1}\rho_1\left(1+\frac{2}{\gamma-1}q\right)^{-1} \tag{1.3}$$
$$p_2 = \frac{2\gamma}{\gamma+1} p_1 \frac{1}{q}\left(1-\frac{\gamma-1}{2\gamma}q\right), \quad \mu_2 = \mu_1, \qquad \left(q = \frac{\gamma p_1}{\rho_1 c_0^2} = \frac{1}{M^2}\right)$$

Here the constant speed c_0 of the shock wave is connected with its radius $r_2(t)$ by the relation $r_2(t) = c_0 t$.

The problem considered here is self-similar [1] only when $\omega = 2$, i.e. when the medium is isothermal. In this case the self-similar variable and the parameters of the gas flow can be written in the form

$$\lambda = r/c_0 t = r/r_2, \quad v = c_0 f(\lambda), \quad \rho = \rho_2 g(\lambda)$$
$$p = p_2 h(\lambda), \quad \mu = \mu_2 e(\lambda)$$
$$\left(\frac{d}{dt}\ln\rho_2 = -\frac{2c_0}{r_2} = \frac{d}{dt}\ln p_2\right) \tag{1.4}$$

Inserting (1.4) into (1.1) and taking into account the relation given above in the parentheses, we have

$$(f-\lambda)\frac{df}{d\lambda} + \frac{\alpha_1}{g}\frac{dh}{d\lambda} + \frac{2qe}{\gamma\lambda^2} = 0 \tag{1.5}$$
$$(f-\lambda)\frac{d}{d\lambda}\ln g + \frac{df}{d\lambda} + \frac{2f}{\lambda} = 2 \tag{1.6}$$
$$(f-\lambda) g\frac{dh}{d\lambda} - \gamma h (f-\lambda)\frac{dg}{d\lambda} + 2gh(\gamma-1) = 0 \tag{1.7}$$
$$\frac{de}{d\lambda} = \frac{\alpha_2\gamma}{2q}\lambda^2 g \tag{1.8}$$

where

$$\alpha_1 = \frac{2(\gamma-1)}{(\gamma+1)^2}\left(1+\frac{2q}{\gamma-1}\right)\left(1-\frac{\gamma-1}{2\gamma}q\right)$$
$$\alpha_2 = \frac{2(\gamma+1)}{\gamma(\gamma-1)} q\left(1+\frac{2}{\gamma-1}q\right)^{-1}$$

Further we write the condition of conservation of mass in a different form: $d\mu / dt = 0$ and we substitute into it (1.4). Eliminating $de/d\lambda$ from the resulting expression and (1.8), we obtain

$$e = \frac{\alpha_2\gamma}{2q}\lambda^2 g(\lambda - f) \tag{1.9}$$

which on substitution into (1.5) yields

$$(f-\lambda)\frac{df}{d\lambda} + \alpha_1\frac{1}{g}\frac{dh}{d\lambda} - \alpha_2(f-\lambda) = 0 \tag{1.10}$$

The conditions (1.3) at the surface of the shock wave on transforming into the dimensionless form and using (1.4), yield

$$f(1) = \frac{2}{\gamma+1}(1-q), \qquad g(1) = 1 \tag{1.11}$$
$$h(1) = 1, \quad e(1) = 1$$

Regarding now (1.6), (1.7) and (1.10) as a system of algebraic equations in $df/d\lambda$,

$dg\,/\,d\lambda$ and $dh\,/\,d\lambda$, we solve it to obtain

$$\frac{df}{d\lambda}=\frac{2\alpha_1(\gamma f/\lambda-1)+\alpha_2 g^2(f-\lambda)^2}{(f-\lambda)^2 g-\alpha_1\gamma h}$$

$$\frac{dg}{d\lambda}=-\frac{g}{\lambda(f-\lambda)}\;\frac{2\alpha_1(\gamma-1)\lambda h+\alpha_2\lambda g^2(f-\lambda)+2g(f-\lambda)^3}{(f-\lambda)^2 g-\alpha_1\gamma h}$$

$$\frac{dh}{d\lambda}=-g\,\frac{h(f-\lambda)(2\gamma f/\lambda-2+\alpha_2\gamma g)}{(f-\lambda)^2 g-\alpha_1\gamma h} \tag{1.12}$$

Numerical solution of the system (1.12) of nonlinear first order differential equations was carried out with the boundary conditions (1.11) when $\gamma={}^5/_3$, for two cases, $M=1.6$ and $M=\sqrt{10}$. The surface of the shock wave at which $\lambda=1$ was used as the starting point for the computation and the Runge-Kutta method was used, followed by that of Milne. The computation was terminated near the value of $\lambda=\lambda_0$ at which $f(\lambda)$ became equal to λ, since $dg/d\lambda$ is infinite at $\lambda=\lambda_0$. The numerical values of f, g, h and e as functions of λ are given in Table 1 for $M=1.6$ and in Table 2 for $M=\sqrt{10}$.

2. Discussion of results. The data given in Tables 1 and 2 show that f, g and h increase with decreasing λ and, when $\lambda\to\lambda_0$, the function h tends to a finite value which is different for different Mach numbers. We note that the density increases faster than velocity, the mass decreases at a greater rate than the Mach number increases and tends to zero as $\lambda\to\lambda_0$. From this it follows that $\lambda=\lambda_0$ represents the inner boundary of the perturbed motion. We also see that λ_0 increases with increasing Mach number.

Comparison of the above results with those obtained from a self-similar motion ($\omega=2.5$) in the presence of a strong explosion investigated by Sedov in [1], shows an agreement between the distributions of the density, mass and pressure, but not of the velocity.

The total energy of the unperturbed medium in a region bounded by the surface of the shock wave at the instant t and the total energy of the perturbed medium bounded from the outside by a spherical shock wave of radius $r_2(t)$ and from the inside by a spherical piston of radius $r_*(t)=\lambda_0 r_2(t)$, are given by

$$E_1=\int_0^{r_2}\left(\frac{p_1}{\gamma-1}-\frac{G\mu_1\rho_1}{r}\right)4\pi r^2\,dr=\frac{8\pi^2A^2G(3-2\gamma)\,r_2(t)}{\gamma-1} \tag{2.1}$$

$$E_2=\int_{r_*}^{r_2}\left(\frac{\rho v^2}{2}+\frac{p}{\gamma-1}-\frac{G\mu\rho}{r}\right)4\pi r^2\,dr \tag{2.2}$$

Substituting the values of v, p, ρ and μ into (2.2) and passing to the dimensionless variable λ, we have in accordance with (1.4),

$$E_2=4\pi^2A^2Gr_2(I_1+I_2-I_3) \tag{2.3}$$

$$I_1=\frac{\alpha_2\gamma^2}{2q^2}\int_{\lambda_0}^{1}f^2g\lambda^2\,d\lambda,\quad I_2=\frac{\alpha_1\alpha_2\gamma^2}{(\gamma-1)q^2}\int_{\lambda_0}^{1}h\lambda^2\,d\lambda$$

$$I_3=\frac{2\alpha_2\gamma}{q}\int_{\lambda_0}^{1}eg\lambda\,d\lambda$$

It can be shown that $g(\lambda)\sim(f-\lambda)^{-4/9}$ and by (1.9), when $\gamma={}^5/_3$, $e(\lambda)\sim(f-\lambda)^{5/9}$

Table 1

$\lambda \cdot 10^2$	$M = 1.6$			
	$f \cdot 10^4$	g	h	$e \cdot 10^4$
100	4570	1.0000	1.0000	10000
98	4618	1.0509	1.0326	9633
96	4669	1.1062	1.0670	9259
94	4723	1.1670	1.1034	8881
92	4781	1.2339	1.1420	8499
90	4843	1.3081	1.1827	8112
88	4909	1.3907	1.2259	7718
86	4978	1.4834	1.2715	7318
84	5052	1.5883	1.3198	6909
82	5131	1.7079	1.3708	6492
80	5214	1.8462	1.4247	6063
78	5302	2.0082	1.4816	5620
76	5396	2.2017	1.5417	5161
74	5496	2.4384	1.6049	4682
72	5603	2.7379	1.6712	4175
70	5717	3.1360	1.7404	3632
68	5839	3.7085	1.8122	3035
66	5970	4.6563	1.8855	2352
64	6113	6.7815	1.9576	1470
63	6188	10.5499	1.9903	0861
62.40	6235	62.2352	2.0058	0200

Table 2

$\lambda \cdot 10^2$	$M = \sqrt{10}$			
	$f \cdot 10^4$	g	h	$e \cdot 10^4$
100	6750	1.0000	1.0000	10000
98	6821	1.0652	1.0198	9378
96	6898	1.1406	1.0405	8740
94	6982	1.2294	1.0622	8083
92	7073	1.3359	1.0848	7401
90	7171	1.4673	1.1082	6688
88	7277	1.6355	1.1320	5934
86	7391	1.8630	1.1562	5123
84	7515	2.1996	1.1801	4225
82	7649	2.7984	1.2031	3189
80	7794	4.4714	1.2237	1810
79	7872	11.6500	1.2316	0626
78.86	7883	35.7368	1.2324	0192

near the spherical piston. Consequently, the last two integrals in (2.3) converge. Substituting $g\lambda^2$ from (1.8) into the first integral of (2.3) and integrating by parts we find, that I_1 also converges. The integrals in (2.3) were computed by the Simpson method, using the value $\lambda_0 = 0.6240$ for the moderately strong shock wave and $\lambda_0 = 0.7886$ for the strong shock wave. As a result, the following energy difference was determined:

$$E_2 - E_1 = C \cdot 4\pi^2 A^2 G r_2, \qquad C = \begin{cases} 2.88, & M = 1.6 \\ 14.8, & M = \sqrt{10} \end{cases}$$

In both cases the rate of change of the energy difference is equal to the work done by the spherical piston given by the formula

$$W = 4\pi p_* r_*^2 dr_* \,/\, dt = 4\pi p_2 h\,(\lambda_0)\, r_2^2 \lambda_0^3 c_0$$

In conclusion we note the following. If the self-similar motion considered here is caused by the release of energy at the instant $t = 0$, then for $\omega = 2$, i.e. for an isothermal medium, the law of energy release discussed by Sedov in [1] assumes the form of a relation proportional to time. Kopal [4] obtained a numerical solution of an analogous problem for a gaseous medium in which the density varied as $r^{-\omega}$. He assumed that the total energy of the perturbed medium bounded by a spherical shock wave is equal to the total energy of an unperturbed medium bounded by a shock wave in the same position at an arbitrary instant, i.e. he neglected the energy of explosion. As the result, Kopal established a relationship connecting the Mach number M with ω and used it to deduce that $M = 1.6$ for $\omega = 2$. However, our computation given above shows that at $M = 1.6$ the energy difference has a finite value and is not equal to zero as was asserted by Kopal.

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